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The Ellipsoid Paradox in Thermodynamics[1]

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A resolution of the "ellipsoid paradox in thermodynamics" is proposed based on elementary geometrical optics.

KEY WORDS: Thermodynamics; geometrical optics; radiative transfer.

Boley and Sculley[(1)] gave an explanation of the ellipsoid paradox by means of a long argument involving quantum statistics. In this note we show that this degree of sophistication is unnecessary and the paradox can be resolved by an argument based on elementary geometrical optics.

Figure 1 shows a perfectly reflecting cavity formed by portions of a sphere and an ellipsoid of revolution. Blackbodies A and B at a common temperature T are placed at the center of the sphere (which is also one focus of the ellipsoid) and at the other focus. The argument of the paradox is as follows: on account of the perfect imaging properties of spheres and ellipsoids most of the radiation from A is reflected back to A but most of the radiation from B is reflected to A, so that at equilibrium A will be hotter than B.

To resolve the paradox it is merely necessary to note that the imaging properties of the ellipsoid are perfect from focus to focus only for the actual foci; however, this image formation does not satisfy the Abbe sine condi-

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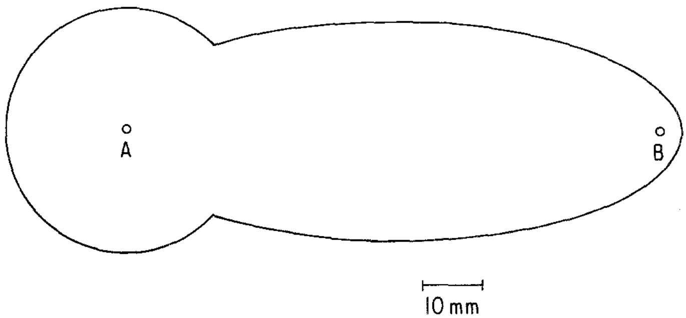

Fig. 1. The ellipsoid paradox: the ellipsoid images B at A "perfectly" and the sphere images A on itself perfectly.

tion[4] so that there is coma for an object point not precisely at the focus. (Coma is a point-imaging aberration which increases in magnitude linearly with distance from the focus.) But if the thermodynamic discussion is to make sense the bodies A and B must have finite heat capacities and therefore finite volumes and thus it follows that on account of the aberration of coma not all the radiation from B reflected from the ellipsoid will reach A. Thus the paradox vanishes.

It may be remarked that since coma increases linearly with distance the above argument holds no matter how small the bodies are made as long as they have a finite size and, of course, diffraction effects would also occur for wavelengths comparable with the size of the objects.

Figure 2 is drawn to approximately the same scale as in the paper by Boley and Sculley, with blackbodies 1 mm in diameter. Several rays from a point on the surface of B 0.5 mm from the axis of the ellipsoid are drawn to scale, showing the magnitude of the effect of the coma.

Yet more physical insight may be gained by considering the converse problem, namely, what shape of reflector surface would send all radiation from a finite object at A onto a similar finite object at B. The solution to this is known if A and B are right circular cylinders;[(3)] it is similar to the

[4] The sine condition was first given by Clausius in 1864 (see Ref. 2) and he derived it from thermodynamical considerations. Abbe's formulation, given in many texts on geometrical optics, is essentially the condition that, if a point B is imaged perfectly at A, the imagery should remain perfect for small displacements of B. The usual formulation is

$$\frac{n' \sin' \alpha}{n \sin \alpha} = \text{const}$$

where α is the angle of a ray with the axis on the object side and α' is the corresponding angle on the image side; n and n' are the respective refractive indices and the ratio must be constant as α varies from zero to its maximum value.

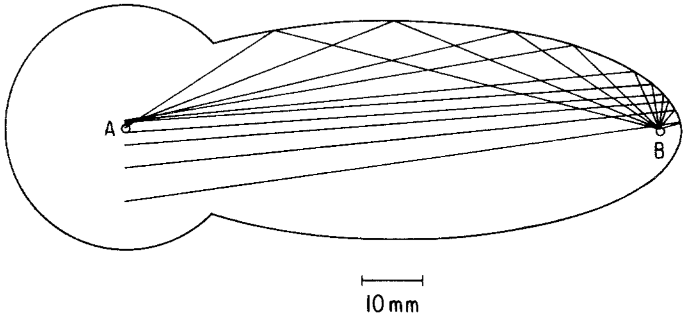

Fig. 2. Resolution of the paradox: points at a finite distance from the focus B are *not* imaged perfectly into the corresponding points near A because of the coma in the image formation.

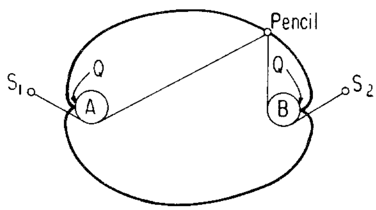

Fig. 3. Construction of nonelliptical reflector for transferring radiation from A onto B. Note that the profile does not even tend to an ellipse as the diameters of A and B tend to zero.

gardener's method of drawing an ellipse and it is illustrated in Fig. 3. The string is anchored at S_1 and S_2 and its length is arranged so that when it is taut the pencil will just reach the point Q at the back of either A or B. The reflector profile is then traced in the usual way. However, this profile does *not* tend to an ellipse of finite eccentricity as the diameters of the bodies A and B tend to zero; it is easily seen that the profile degenerates to two superimposed straight lines joining A and B; this is, strictly, a degenerate ellipse, but since it encloses zero area no radiation can flow through it. Thus it can be seen that an elliptical shape is not the limiting shape obtained as the two blackbodies become infinitely small, so that there is no reason to suppose that all the radiation from one could be transferred to the other.

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